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Ultradiscrete QRT maps and tropical elliptic curves

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Abstract
It is shown that the polygonal invariant curve of the ultradiscrete QRT (uQRT) map, which is a two-dimensional piecewise linear integrable map, is the complement of the tentacles of a tropical elliptic curve on which the curve has a group structure in analogy to classical elliptic curves. Through the addition formula of a tropical elliptic curve, a tropical geometric description of the uQRT map is then presented. This is a natural tropicalization of the geometry of the QRT map found by Tsuda. Moreover, the uQRT map is linearized on the tropical Jacobian of the corresponding tropical elliptic curve in terms of the Abel–Jacobi map. Finally, a formula concerning the period of a point in the uQRT map is given, and an exact solution to its initial-value problem is constructed by using the ultradiscrete elliptic theta function.

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1. Introduction

The QRT maps introduced by Quispel, Roberts and Thompson in 1989 [1] form an 18-parameter family of two-dimensional paradigmatic integrable maps. They include reductions of various higher-dimensional soliton systems such as the KdV, the modified KdV and the nonlinear Schrödinger equations, and each member possesses a one-parameter family of invariant curves which fills the plane and is parametrized by elliptic functions. Though the QRT maps were introduced by using purely algebraic relations of vectors, Tsuda revealed its geometry in terms of the addition formula of a rational elliptic surface [2]. His geometric formulation made it clear that the QRT map is an autonomous limit of Sakai's elliptic Painlevé equation [3].

In the 1990s, the study of soliton systems has received a substantial boost with the introduction of the ultradiscretization procedure which allows a systematic construction of cellular automata from soliton systems [4–6]. The method was immediately applied to the QRT maps, and an eight-parameter family of piecewise linear maps called the ultradiscrete

QRT (uQRT) maps was derived [7, 8]. Each member possesses a family of polygonal invariant curves which fills the plane and is parametrized by the ultradiscrete elliptic functions [9]. Though some attempts to give a geometric description of the uQRT maps have been made [10, 11], its geometry has not been clarified yet.

Recently, Inoue and Takenawa proposed a method [12] to study integrable cellular automata via the tropical spectral curve and its Jacobian [13]. They applied the method to the periodic box-ball system [14] and clarified the algebro-geometrical meaning of the real torus introduced for its initial-value problem [15–17]. This fact suggests that a tropical geometric approach is effective to examine ultradiscrete integrable systems.

In this paper, we give a geometric description of the uQRT maps via tropical elliptic curves. We at first show that the invariant curve of the uQRT map is the complement of the tentacles of a tropical elliptic curve on which the curve has a group structure [18]. We then present a geometric description of the uQRT map in terms of the addition formula of a tropical elliptic curve. Through this description, we linearize the uQRT map on the tropical Jacobian of the corresponding tropical elliptic curve by using the Abel–Jacobi map. Finally, we give a formula concerning the fundamental period of a point in the uQRT map, and construct an exact solution to its initial-value problem by using the ultradiscrete elliptic theta function.

2. The uQRT maps and tropical elliptic curves

2.1. The uQRT maps

We consider an eight-parameter family of piecewise linear maps $\phi : (x, y) \mapsto (\bar{x}, \bar{y})$ called the uQRT maps [7, 8]:

$$\bar{x} = F_1(y) - F_3(y) - x, \qquad \bar{y} = G_1(\bar{x}) - G_3(\bar{x}) - y, \tag{1}$$

where we put

$$F_1(y) := \max[\alpha_{20} + 2y, \alpha_{21} + y, \alpha_{22}], \qquad F_3(y) := \max[\alpha_{00} + 2y, \alpha_{01} + y, \alpha_{02}],$$
$$G_1(\bar{x}) := \max[\alpha_{02} + 2\bar{x}, \alpha_{12} + \bar{x}, \alpha_{22}], \qquad G_3(\bar{x}) := \max[\alpha_{00} + 2\bar{x}, \alpha_{10} + \bar{x}, \alpha_{20}].$$

Each member of this family possesses a one-parameter family of invariant curves filling the plane

$$k + H(B_0; x, y) = H(A_0; x, y), \tag{2}$$

where we put

$$H(A_0; x, y) := \max[F_1(y), F_2(y) + x, F_3(y) + 2x]$$
$$H(B_0; x, y) := x + y$$

and $F_2(y) := \max[\alpha_{10} + 2y, \alpha_{12}]$. The family of invariant curves is denoted by I_k ($k \in \mathbb{R}$).

Example 1. Let the parameters be as follows:

$$\alpha_{00} = -5, \qquad \alpha_{01} = 10, \qquad \alpha_{02} = 0, \qquad \alpha_{10} = 10,$$
$$\alpha_{12} = 5, \qquad \alpha_{20} = 0, \qquad \alpha_{21} = 5, \qquad \alpha_{22} = 0.$$

Then I_{14} is a pentagonal, I_{24} is a heptagonal and I_{30} is a octagonal invariant curve of ϕ (see figure 1). In figure 1, a parameter α_{ij} ($0 \leqslant i, j \leqslant 2$) stands for a domain in which an edge of I_k is represented by the linear equation $k = \alpha_{ij} + (1 - i)x + (1 - j)y$.

If $\alpha_{00} > \alpha_{01} + \alpha_{10} - k$ holds then I_k passes through the domain in which I_k is represented by $k = \alpha_{00} + x + y$ (see I_{30} in figure 1). Then I_k corresponds not to a tropical elliptic curve but to a tropical curve of degree 4 (see section 2.2). In order to establish a connection between the family of invariant curves filling the plane and a family of tropical elliptic curves, we assume $\alpha_{00} \to -\infty$ in the following.

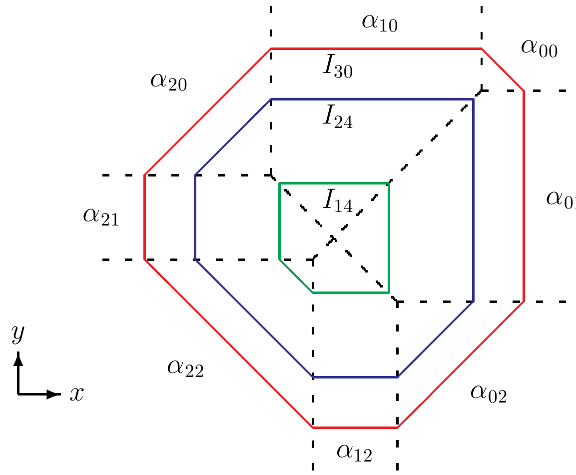


Figure 1. Polygonal invariant curves of the uQRT map.

2.2. *Tropical elliptic curves*

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ be the tropical semifield [19]. Consider the following tropical polynomial f in (x, y) with support $\mathcal{A} = \{(a_1, a_2) \in \mathbb{Z}^2 | 0 \leqslant a_1, a_2 \leqslant 2, a_1 + a_2 \leqslant 3\}$,

$$f = \max_{(a_1,a_2)\in\mathcal{A}} [\lambda_{(a_1,a_2)} + a_1 x + a_2 y], \tag{3}$$

where each $\lambda_{(a_1,a_2)} \in \mathbb{T}$. The tropical curve given by f is the set of all points $(x, y) \in \mathbb{R}^2$ at which f is not smooth [19, 20]. The lower face of the convex hull of $\{(a_1, a_2, \lambda_{(a_1,a_2)}) | (a_1, a_2) \in \mathcal{A}\}$ projects bijectively onto that of $\mathcal{A}$ under deleting the last coordinate. This defines a regular subdivision Δ of $\mathcal{A}$.

We have the following duality theorem [19].

Theorem 1 (proposition 3.5 in [19])**.** *The tropical curve C given by a tropical polynomial is an embedded graph in $\mathbb{R}^2$ which is dual to the regular subdivision Δ of the support of the tropical polynomial. Corresponding edges of C and Δ are perpendicular.*

The tropical curve C given by (3) has degree 3 [18] and genus one because Δ contains an interior lattice point $(1, 1)$ [18, 21]. Assume

$$2\lambda_{(1,1)} > \max[\lambda_{(2,0)} + \lambda_{(0,2)}, \lambda_{(2,1)} + \lambda_{(0,1)}, \lambda_{(1,2)} + \lambda_{(1,0)}].$$

Then C is smooth (i.e. every vertex is trivalent with multiplicity one) [21, 18], and hence it is a tropical elliptic curve [18].

Since $(1, 1)$ is an interior lattice point of Δ, C contains a unique cycle, which we will denote by $\bar{C}$. We call each connected component of $C \backslash \bar{C}$ a tentacle of C.

Example 2. Let the parameters be as follows:

$$\begin{array}{llll} \lambda_{(2,1)} = 10, & \lambda_{(2,0)} = 0, & \lambda_{(1,2)} = 10, & \lambda_{(1,0)} = 5, \\ \lambda_{(1,1)} = 20, & \lambda_{(0,2)} = 0, & \lambda_{(0,1)} = 5, & \lambda_{(0,0)} = 0. \end{array} \tag{4}$$

Then C is a heptagon with seven half rays called tentacles. Figure 2 shows C and Δ. We see that each edge of C is perpendicular to an edge of Δ.

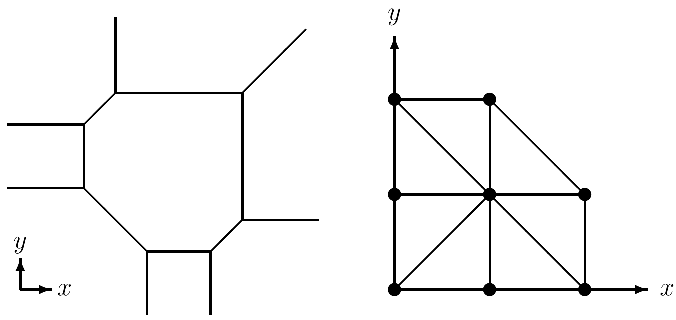


Figure 2. A tropical elliptic curve and the regular subdivision.

2.3. The group law on a tropical elliptic curve

In [18], Vigeland showed that tropical elliptic curves have a group structure in analogy to classical elliptic curves. In this section we recall some material from his (and related) work, which will be crucial in the following.

We define stable intersection $C \cap_{st} D$ of two tropical curves C and D as follows:

$$C \cap_{st} D = \lim_{\epsilon \to 0} (C_\epsilon \cap D_\epsilon),$$

where C_ϵ and D_ϵ are nearly translations of C and D such that they intersect transversally (i.e. no vertex of C_ϵ lies on D_ϵ and vice versa) [19].

Fix $\mathcal{O} \in \bar{C}$. Let $V_1, V_2, \ldots, V_n$ be the vertices of $\bar{C}$ in counterclockwise direction. If $\mathcal{O}$ is a vertex then $V_1 = \mathcal{O}$, otherwise $\mathcal{O}$ lies between V_1 and V_n. The edge connecting V_i and V_{i+1} is denoted by E_i for $i = 1, 2, \ldots, n-1$. E_n is the edge connecting V_n and V_1. For $i = 1, 2, \ldots, n$, let $\varepsilon_i = 1/|\boldsymbol{v}_i|$, where $\boldsymbol{v}_i$ is the primitive tangent vector along E_i and $|\boldsymbol{v}_i|$ denotes its Euclidian length. The total lattice length $\mathcal{L}$ of $\bar{C}$ is defined as

$$\mathcal{L} = \sum_{i=1}^{n} \varepsilon_i |E_i|.$$

Then the tropical Jacobian $J(\bar{C})$ of $\bar{C}$ is defined as follows [12, 13]:

$$J(\bar{C}) = \mathbb{R}/\mathcal{L}\mathbb{Z}.$$

Now we define a map $\eta : \bar{C} \to J(\bar{C})$, which is linear on each edge E_i of $\bar{C}$:

$$\begin{aligned} &\eta(\mathcal{O}) = 0, \\ &\eta(V_1) = \varepsilon_n |\mathcal{O}V_1|, \\ &\eta(V_{i+1}) = \eta(V_i) + \varepsilon_i |E_i|, \qquad (i = 1, 2, \ldots, n-1). \end{aligned} \tag{5}$$

Note that the map (5) is bijective and equivalent to the Abel–Jacobi map (2.3) for $g = 1$ in [12]. We define the single lattice distance $d_C(P, Q)$ between points P and Q on $\bar{C}$ by

$$d_C(P, Q) = \eta(Q) - \eta(P).$$

The following theorem is the main result of [18].

Theorem 2 (theorem 1.1 in [18])**.** *Let C be a tropical elliptic curve and $\mathcal{O}$ a point on $\bar{C}$.*

(i) We have a bijection of sets $\bar{C} \leftrightarrow \mathrm{Pic}^0(C)$[1], given by $P \leftrightarrow P - \mathcal{O}$.

[1] In [18], this group of degree 0 divisor classes on C [22] is called the Jacobian of C and denoted by $\mathrm{Jac}(C)$.

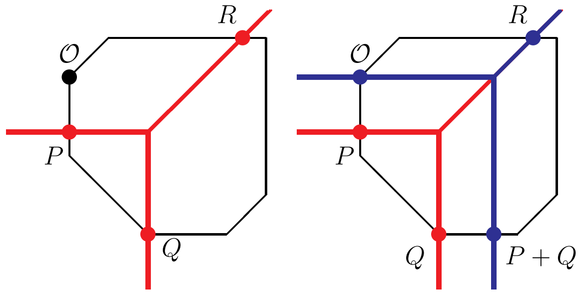


Figure 3. The group law on a tropical elliptic curve.

(ii) The induced group law on $\bar{C}$ satisfies the relation

$$d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q).$$

(iii) As a group, $\bar{C}$ is isomorphic to S^1.

We can describe the group law geometrically. Given two points P and Q on $\bar{C}$, we cannot always find a tropical line L (i.e. a tropical curve of degree 1) which intersects C stably in P and Q. If there exists such a line, we call (P, Q) a good pair. If (P, Q) is a good pair, consider L through both P and Q, and let R be the third intersection point of L and $\bar{C}$. Now if $(R, \mathcal{O})$ is a good pair, let L' be the tropical line through both R and $\mathcal{O}$. Then $P + Q$ is the third intersection point of L' and $\bar{C}$ (see figure 3).

If any of the pairs (P, Q) and $(R, \mathcal{O})$ fails to be good, then move the two points involved equally far in the lattice metric in opposite directions until they form a good pair, and use this new pair as described above.

3. Tropical geometric description of the uQRT maps

3.1. Geometry of the uQRT maps

Now let C be a tropical elliptic curve given by (3). We first establish a correspondence between I_k and $\bar{C}$.

Lemma 1. *The invariant curve I_k coincides with $\bar{C}$ with the choice of the parameters*

$$\begin{aligned} &\alpha_{ij} = \lambda_{(2-i,2-j)}, \qquad (0 \leqslant i, j \leqslant 2, (i, j) \neq (0, 0), (1, 1)),\\ &k = \lambda_{(1,1)}. \end{aligned} \tag{6}$$

Proof. Each edge of I_k is given by one of the following seven linear equations

$$k = \alpha_{ij} + (1 - i)x + (1 - j)y,$$

for $0 \leqslant i, j \leqslant 2$ and $(i, j) \neq (0, 0), (1, 1)$. If we choose the parameters as in (6) we have

$$\lambda_{(1,1)} = \lambda_{(2-i,2-j)} + (1 - i)x + (1 - j)y,$$

for $0 \leqslant i, j \leqslant 2$ and $(i, j) \neq (0, 0), (1, 1)$. By setting $l = 2 - i$ and $m = 2 - j$, this reduces to

$$\lambda_{(1,1)} + x + y = \lambda_{(l,m)} + lx + my \tag{7}$$

for $0 \leqslant l, m \leqslant 2$ and $l + m \leqslant 3$. Each linear equation (7) gives a part of C, which is not a tentacle but a line segment of $\bar{C}$. Because the left-hand side of (7) is a term of f corresponding

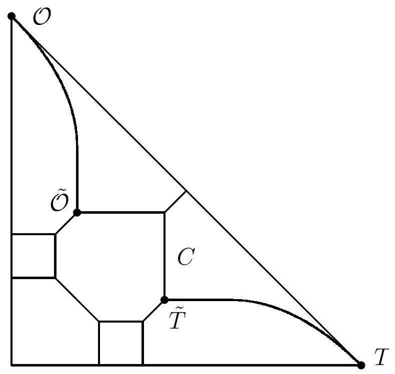


Figure 4. A tropical elliptic curve C in $\mathbb{P}^{2,\text{trop}}$.

to the only interior lattice point $(1, 1)$ of Δ, the part of C given by (7) is perpendicular to an edge outgoing from $(1, 1)$. Thus I_k coincides with $\bar{C}$. $\square$

In order to see the correspondence between a tropical and a classical elliptic curve, we consider tropical elliptic curves in the tropical projective plane $\mathbb{P}^{2,\text{trop}}$ introduced by Kajiwara [23][2]. The tropical projective plane $\mathbb{P}^{2,\text{trop}}$ is a two-dimensional real space with three boundaries corresponding to the lines at infinity $X = 0$, $Y = 0$ and $Z = 0$, where $[X, Y, Z]$ are the homogeneous coordinates of $\mathbb{P}^2$, respectively [23].

Example 3. Figure 4 shows C in $\mathbb{P}^{2,\text{trop}}$ with the choice of the parameters (4). Tentacles having the primitive tangent vectors $(1, 0)$ and $(0, 1)$ pass through T and $\mathcal{O}$ corresponding to the points at infinity $[1, 0, 0]$ and $[0, 1, 0]$ in $\mathbb{P}^2$, respectively [23].

Now we show that every C passes through both T and $\mathcal{O}$. Let D_1 and D_2 be sets of vectors as follows:

$$D_1 := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,\middle|\, -\frac{\pi}{2} < \theta < \frac{\pi}{4}, r \in \mathbb{R}_{>0} \right\},$$

$$D_2 := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,\middle|\, \frac{\pi}{4} < \theta < \pi, r \in \mathbb{R}_{>0} \right\}.$$

A tentacle whose primitive tangent vector is in D_1 passes through T and in D_2 through $\mathcal{O}$ [23]. We have the following lemma.

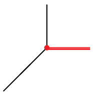
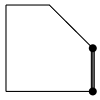
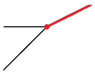
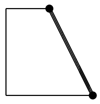
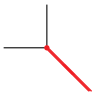
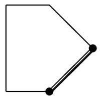
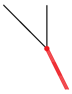
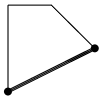
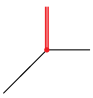
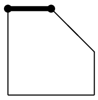
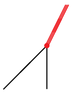
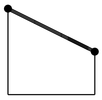
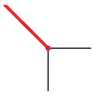
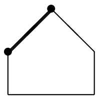
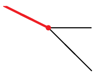
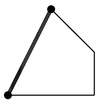
Lemma 2. *There exists exactly one tentacle of C whose primitive tangent vector is in D_i for $i = 1, 2$.*

Proof. Note first that a tentacle of C is perpendicular to an edge of Δ not through $a_{11} := (1, 1)$. Also note that we can eliminate any vertex of Δ by setting $\lambda_{(i,j)} = -\infty$ for some $(i, j) \in \mathcal{A}$.

In order C to be genus one, a_{11} must be the interior point of Δ. Therefore, the vertices $a_{20} := (2, 0)$ and $a_{21} := (2, 1)$ of Δ must not vanish simultaneously. If both a_{20} and a_{21} exist, there exists a tentacle perpendicular to the edge connecting them. The primitive tangent vector of this tentacle is $(1, 0) \in D_1$. It is easy to see that no other tentacle has the primitive tangent vector in D_1. Suppose a_{21} to be eliminated by setting $\lambda_{(2,1)} = -\infty$. Then a_{20} and $a_{12} := (1, 2)$ do not vanish simultaneously, and hence there exists a tentacle whose primitive tangent vector

[2] $\mathbb{P}^{2,\text{trop}}$ is equivalent not to $\mathbb{TP}^2 := \mathbb{R}^3/(1, 1, 1)$ in [19] but to $\mathbb{TP}^2$ in [13].

Table 1. The first column shows C around a tentacle whose primitive tangent vector is given in the second column. The third column shows a parameter taking $-\infty$ to eliminate a vertex of Δ. The last column shows an edge of Δ perpendicular to the tentacle.

Tentacle	Vector	$-\infty$	Edge in Δ
	$(1, 0) \in D_1$	None	
	$(2, 1) \in D_1$	$\lambda_{(2,1)}$	
	$(1, -1) \in D_1$	$\lambda_{(2,0)}$	
	$(1, -2) \in D_1$	$\lambda_{(2,0)}, \lambda_{(1,0)}$	
	$(0, 1) \in D_2$	None	
	$(1, 2) \in D_2$	$\lambda_{(1,2)}$	
	$(-1, 1) \in D_2$	$\lambda_{(0,2)}$	
	$(-2, 1) \in D_2$	$\lambda_{(0,1)}, \lambda_{(0,2)}$	

is $(2, 1) \in D_1$. Similarly, if a_{20} vanishes then there exists a tentacle whose primitive tangent vector is $(1, -1) \in D_1$ or $(1, -2) \in D_1$. Thus, for any choice of the parameters, there exists exactly one tentacle whose primitive tangent vector is in D_1. In a similar manner, the D_2 case is shown. We summarize this in table 1. $\square$

Let $\tilde{T}$ and $\tilde{\mathcal{O}}$ be the vertices of $\bar{C}$ connected with T and $\mathcal{O}$ by the tentacles whose primitive tangent vectors are in D_1 and D_2, respectively. Then $\tilde{T}$ and $\tilde{\mathcal{O}}$ are linearly equivalent to T and $\mathcal{O}$, respectively (see figure 4) [18].

Now we show that the uQRT map is nothing but the addition of points on a tropical elliptic curve. Let $P = (x, y)$ be a point on I_k. By lemma 1, P can be identified with a point on $\bar{C}$. Note that $\bar{C}$ has at most two line segments parallel to the x-axis, we denote them by $\mathcal{E}_1$ and $\mathcal{E}_2$. Assume P is not on $\mathcal{E}_1$ and $\mathcal{E}_2$, then $(P, \tilde{T})$ is a good pair. Let us consider the line through both P and $\tilde{T}$, and denote it by L_1. By the tropical Bézout theorem [19, 23], L_1 intersects $\bar{C}$ at three points. We denote the third intersection point of L_1 and $\bar{C}$ by $Q = (\bar{x}, y)$. Note that

P and Q are on the x-ray of L_1 (i.e. the half ray parallel to the x-axis), and hence Q is not on $\mathcal{E}_1$ and $\mathcal{E}_2$.

The coordinate $\bar{x}$ can be written in x and y as follows. Since I_k passes through both P and Q, we have

$$H(A_0; x, y) = k + H(B_0; x, y), \qquad H(A_0; \bar{x}, y) = k + H(B_0; \bar{x}, y).$$

Eliminating k, we get

$$H(A_0; x, y) + \bar{x} = H(A_0; \bar{x}, y) + x. \tag{8}$$

Since P and Q are not on $\mathcal{E}_1$ and $\mathcal{E}_2$, we have

$$H(A_0; x, y) = \max[F_1(y), F_2(y) + x, F_3(y) + 2x] \neq F_2(y) + x,$$
$$H(A_0; \bar{x}, y) = \max[F_1(y), F_2(y) + \bar{x}, F_3(y) + 2\bar{x}] \neq F_2(y) + \bar{x}.$$

On the other hand, P and Q are on the x-ray of L_1, hence we have

$$H(A_0; x, y) = F_1(y) \Rightarrow H(A_0; \bar{x}, y) = F_3(y) + 2\bar{x},$$
$$H(A_0; x, y) = F_3(y) + 2x \Rightarrow H(A_0; \bar{x}, y) = F_1(y).$$

Therefore, (8) can be solved

$$F_1(y) + \bar{x} = F_3(y) + 2\bar{x} + x \qquad \text{or} \qquad F_3(y) + 2x + \bar{x} = F_1(y) + x.$$

In both cases, we obtain

$$Q = (\bar{x}, y) = (F_1(y) - F_3(y) - x, y).$$

Next we consider the case when P is on $\mathcal{E}_1$ or $\mathcal{E}_2$. If we translate P to the point $P' = (x' := x - \zeta, y)$ on the nearest vertex in the left direction, then $(P', \tilde{T})$ is a good pair. As above, let us consider the line L_1' through both P' and $\tilde{T}$, and let the third intersection point of L_1' and $\bar{C}$ be $Q' = (\bar{x}', y)$. Then we have

$$\bar{x}' = F_1(y) - F_3(y) - (x - \zeta).$$

Let $\tilde{T}'$ be the point which is the translation of $\tilde{T}$ by ζ in the lattice metric in opposite direction from P to P' (see figure 5). Now we translate L_1' by ζ in the left direction along the x-axis, and denote the translated line by L_1. Then L_1 passes through P' because P' is on the x-ray of L_1'. By the definition of $\tilde{T}'$, it is clear that L_1 passes through $\tilde{T}'$. Moreover, L_1 intersects $\bar{C}$ stably in P' and $\tilde{T}'$. Because L_1' intersects $\bar{C}$ stably in P' and $\tilde{T}$, a translation of L_1' also intersects $\bar{C}$ stably in P' and a translation of $\tilde{T}$ (see figure 5). Therefore $(P', \tilde{T}')$ is a good pair. The third intersection point Q of L_1 and $\bar{C}$ is

$$Q = (\bar{x}' - \zeta, y) = (F_1(y) - F_3(y) - x, y).$$

In a similar manner, let L_2 be the tropical line through both Q and $\tilde{O}$, then the third intersection point of L_2 and $\bar{C}$ is $\bar{P} = (\bar{x}, \bar{y})$, where $\bar{y}$ is given as

$$\bar{y} = G_1(x') - G_3(x') - y.$$

The correspondence $P \mapsto \bar{P}$ is nothing but the uQRT map ϕ.

From the above discussion, it is clear that the correspondence $P \mapsto \bar{P}$ can also be described as the addition of points on $\bar{C}$ (see figure 6)

$$P + \tilde{T} = \bar{P} + \tilde{O} = \bar{P}.$$

Thus we obtain the following theorem.

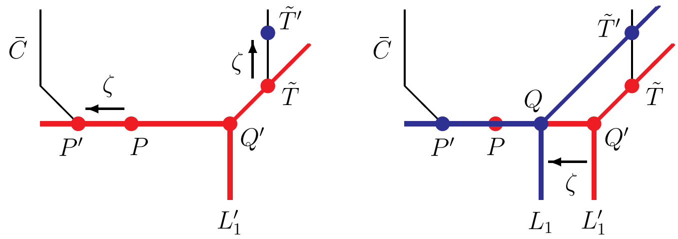


Figure 5. The case when P is on $\mathcal{E}_1$ or $\mathcal{E}_2$.

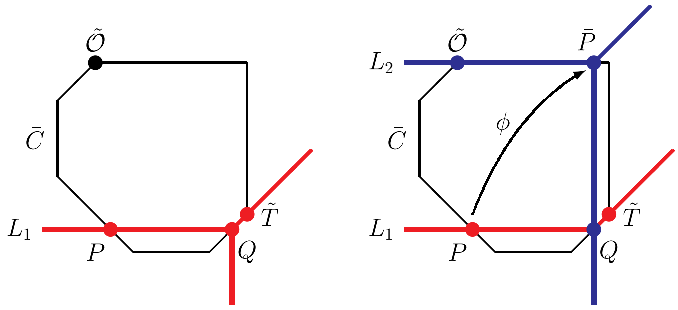


Figure 6. The uQRT map $\phi : P \mapsto \bar{P}$.

Theorem 3. *Let P be a point in $\mathbb{R}^2$ and I_k the invariant curve of the uQRT map (1) with the initial value P. Let C be the tropical elliptic curve given by (3) with (6). Then the uQRT map $\phi : P \mapsto \bar{P}$ is equivalent to the addition formula of $\bar{C}$ (see figure 6),*

$$P + \tilde{T} = \bar{P} + \tilde{O} = \bar{P}.$$

Corollary 1. *The uQRT map $\phi : P \mapsto \bar{P}$ is linearized on $J(\bar{C})$ in terms of (5),*

$$\eta(P) \mapsto \eta(P) + \eta(\tilde{T}).$$

3.2. Fundamental periods

Now we consider the period of a point in the uQRT map. Let $P_0 \in \mathbb{R}^2$ and

$$P_0 \overset{\phi}{\mapsto} P_1 \overset{\phi}{\mapsto} \cdots \overset{\phi}{\mapsto} P_n \overset{\phi}{\mapsto} \cdots.$$

The smallest $n \in \mathbb{N}$ satisfying $P_n = P_0$ is called the fundamental period of P_0 in ϕ.

By theorems 2 and 3, the single lattice distance $d_C(\tilde{O}, P_n)$ between $\tilde{O}$ and P_n on $\bar{C}$ is inductively computed as follows:

$$\begin{aligned} d_C(\tilde{O}, P_n) &= d_C(\tilde{O}, P_{n-1} + \tilde{T}) \\ &= d_C(\tilde{O}, P_{n-1}) + d_C(\tilde{O}, \tilde{T}) \\ &= \cdots \\ &= d_C(\tilde{O}, P_0) + n \times d_C(\tilde{O}, \tilde{T}). \end{aligned}$$

This can be represented by the Abel–Jacobi map (5)

$$\eta(P_n) = \eta(P_0) + n \times \eta(\tilde{T}).$$

Since η is bijective, the initial-value problem for the uQRT map is solved,

$$P_n = \eta^{-1}(\eta(P_0) + n \times \eta(\tilde{T})).$$

If $P_n = P_0$ we have

$$n \times \eta(\tilde{T}) \equiv 0 \pmod{\mathcal{L}\mathbb{Z}}.$$

Thus we have the following theorem.

Theorem 4. *The fundamental period of P_0 in ϕ is*

$$\frac{\mathcal{L}}{\gcd(\eta(\tilde{T}), \mathcal{L})}.$$

Example 4. Let the parameters $\alpha_{01}, \ldots, \alpha_{22}$ be as in example 1. Let $P_0 = (4, 4)$. Then the conserved quantity of ϕ is $k = 14$, and I_{14} is a pentagon with the following five vertices (see figure 1):

$$V_1 = \tilde{O} = (-9, 4), \qquad V_2 = (-9, -5), \qquad V_3 = (-5, -9),$$
$$V_4 = \tilde{T} = (4, -9), \qquad V_5 = (4, 4).$$

Hence C satisfying $\bar{C} \simeq I_{14}$ is uniquely determined. The total lattice length is $\mathcal{L} = 48$. By using these data, we can compute the value of η at $\tilde{T} = (4, -9)$ as $\eta(\tilde{T}) = 22$. Thus the fundamental period of $P_0 = (4, 4)$ in ϕ is

$$\frac{48}{\gcd(22, 48)} = 24.$$

If we choose the initial value as $P_0 = (-19, 0)$ we obtain $k = 24$ and the heptagonal invariant curve I_{24} with the following seven vertices (see figure 1):

$$V_1 = \tilde{O} = (-10, 14), \qquad V_2 = (-19, 5), \qquad V_3 = (-19, -5)$$
$$V_4 = (-5, -19), \qquad V_5 = (5, -19), \qquad V_6 = \tilde{T} = (14, -10), \qquad V_7 = (14, 14).$$

Therefore, we have $\mathcal{L} = 100$ and $\eta(\tilde{T}) = 52$. Thus the fundamental period of $P_0 = (-19, 0)$ in ϕ is

$$\frac{100}{\gcd(52, 100)} = 25.$$

3.3. Solutions

Exact solutions to the initial-value problem for the uQRT map $\phi : (x(t), y(t)) \mapsto (x(t+1), y(t+1))$ is given as follows:

$$x(t) = \rho_1\{H_1(gt) - H_1(gt - \omega_1)\} + \delta_1,$$
$$y(t) = \rho_2\{H_2(gt) - H_2(gt - \omega_2)\} + \delta_2,$$

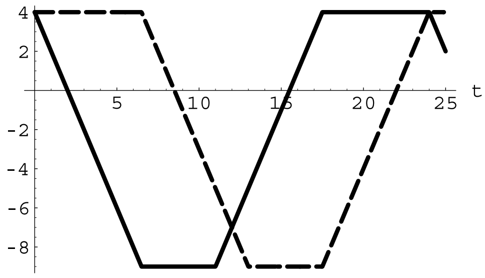
where we put $g := \gcd(\eta(\tilde{T}), \mathcal{L})$ and

$$H_i(u) := \Theta_0\left(\frac{u + \xi_i}{\mathcal{L}}\right) - \Theta_0\left(\frac{u + \xi_i - \gamma_i}{\mathcal{L}}\right)$$

for $i = 1, 2$, and $\Theta_0(u)$ is the ultradiscrete elliptic theta function [9, 24],

$$\Theta_0(u) := -u^2 + \max_{n=-\infty}^{\infty} [2nu - n^2].$$

The parameters $\gamma_i, \delta_i, \rho_i, \omega_i, \xi_i$ for $i = 1, 2$ are uniquely determined by the initial value $(x(0), y(0))$ via the data of $\bar{C}$.



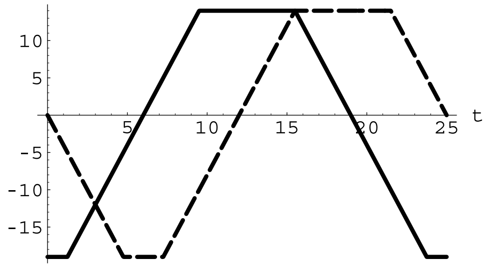


Figure 7. Solutions to the uQRT map.

Example 5. We consider the same situation as example 4. Put $(x(0), y(0)) = (4, 4)$, then we have $\mathcal{L} = 48$ and $\eta(\tilde{T}) = 22$. Since $\bar{C}$ is symmetric under the reflection with respect to $y = x$, we have

$$\gamma_1 = \gamma_2 = \frac{\eta(V_5) - \eta(V_4) - \eta(V_2) + \eta(V_1) + \mathcal{L}}{2} = 26,$$

$$\omega_1 = \omega_2 = \frac{\mathcal{L} - \eta(V_5) + \eta(V_4) - \eta(V_2) + \eta(V_1)}{2} = 13,$$

$$\rho_1 = \rho_2 = \frac{\mathcal{L}}{2\omega_1}(\eta(V_4) - \eta(V_2)) = 24,$$

$$\delta_1 = \delta_2 = -9 + \rho_1 \frac{2\gamma_1\omega_1}{\mathcal{L}^2} = -\frac{47}{24}.$$

The parameters ξ_1 and ξ_2, which determine the initial phase, are given as follows:

$$\xi_1 = 2, \qquad \xi_2 = 37.$$

On the other hand, for $(x(0), y(0)) = (-19, 0)$, we obtain $\mathcal{L} = 100$ and $\eta(\tilde{T}) = 52$, and the parameters can also be computed by using the data of $\bar{C}$:

$$\gamma_1 = \gamma_2 = 57, \qquad \omega_1 = \omega_2 = 33, \qquad \rho_1 = \rho_2 = 50$$

$$\delta_1 = \delta_2 = -\tfrac{19}{100}, \qquad \xi_1 = 45, \qquad \xi_2 = 21.$$

Figure 7 shows the solutions $x(t)$ (solid lines) and $y(t)$ (broken lines) to ϕ with the initial values $(x(0), y(0)) = (4, 4)$ (left) and $(x(0), y(0)) = (-19, 0)$ (right), respectively.

4. Concluding remarks

We present a geometric description of the uQRT map in terms of the group law of a tropical elliptic curve through the correspondence between the invariant curve of the uQRT map and the complement of the tentacles of the tropical elliptic curve on which the curve has the group structure. Using the Abel–Jacobi map, we linearize the uQRT map on the tropical Jacobian of the corresponding tropical elliptic curve. We then solve the initial-value problem for the uQRT map and give a formula concerning the fundamental period of a point in the map. An exact solution to its initial-value problem is given by using the ultradiscrete elliptic theta function.

It is well known that the algebro-geometric approach is effective to examine classical integrable systems [25]. Recent development of tropical geometric methods via tropical spectral curves and their Jacobians [12, 13] and our result have shed light on the contribution of tropical geometry to ultradiscrete integrable systems. Given our analysis, we believe this contribution has merit in the study of other ultradiscrete systems. In the future, we intend to broaden tropical geometric methods to study various ultradiscrete systems, e.g. ultradiscrete analogues of Sakai's elliptic Painlevé equations [3] and the generalized QRT map [2] associated with a higher-dimensional projective variety.

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